Two and three-dimensional computation of dispersion curves of ultrasonic guided waves in isotropic plates by the spectral collocation method

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Abstract :

This paper presents a numerical approache used for plotting the dispersion curves of cross-section ultrasonic guided waves. The spectral collocation method (SCM) described here can turn the set of partial differential equations for sound waves into an eigenvalue problem. In order to evaluate the efficiency of this method for an isotropic aluminum plate, we have established algorithm excuted with Matlab program. The results were compared with a classical bisection zero-finding method, the stiffness matrix method, and SAFE method. The results found confirm that the SCM remains conceptually simpler and depends on the differentiation matrices used. Finally, the method prove its accuracy, its calculation speed and its capacity to compute the phase velocity and wavenumber curves as well as the complete three-dimensional dispersion spectrum which includes both propagative (real wavenumber) and non-propagative modes (complex wave number).

Keywords : Dispersion curves, Spectral Collocation Method, Lamb waves, Guided waves.

1 Introduction

Lamb waves are widely used for non-destructive evaluation (NDE) of finite-dimensional solids. They were first described in 1917 by the English mathematician Sir Horace Lamb (1849–1934) [1]. They belong to a type of waves found in thin plates without traction and shell structures. These waves are used to detect damage inside or on both surfaces of the plates.

In the literature, the modes have been categorized in several ways, we use the classification of Auld [2]: the modes of propagation have a real wavenumber, that's why they are often the most useful for the applications of engineering in CND since they propagate and transport energy inside the structure without attenuation. The second type of solution is that of non-propagating modes with a complex or purely imaginary wavenumber.

Dispersion curves are therefore essential since they make it possible to know the phase and propagation speeds of the waves (group speed) depending on the frequency of the wave generated as well as the thickness of the plate that we wants to inspect. In previous research, authors have discussed dispersion curves and their multiple applications in NDT. They proposed several techniques for drawing the dispersion curves. These methods include bisection method [3], Newton-Raphson method [4], transfer matrix method [5], etc.

To understand well the behavior of Lamb waves, the FEM remains the most robust numerical tool available. However, such simulations in the FEM still require a lot of computational resources even with the current computing power. These facts have prompted many researches to develop other types of numerical methods. i.e. finite difference method [6], spectral element method [7], hybrid boundary element (HBE) method [8], wave finite element (WFE) method [9] as well as the semi-analytical finite element method (SAFE) [10].

The spectral collocation method (SCM), [11], has been generalized as an alternative to classical partial wave root finding (PWRF) routines. To solve elastic (lossless) guided wave problems. Spectral methods were introduced in the 1970s in the field of fluid dynamics by Kreiss and Oliger [12] and have remained a standard computational tool in the field ever since. Using the spectral method, the governing differential equations of GUW are first reduced to the ordinary differential equations over the thickness of the plate. This domain is then discretized into a set of collocation points, thus, we obtain an approximation of the governing equations by an eigenvalue matrix problem.

The main objective of this study is to evaluate the performance of the SCM in terms of accuracy and computational cost for plotting dispersion curves for an isotropic material. To verify the accuracy of the method, we compared the curves plotted with those plotted by the Dispersion Calculator software which uses the stiffness matrix method SMM [13] and with the SAFE method. We first present the theoretical formulation of the method for an arbitrary section waveguide, then, we discuss some parameters that need to be optimized to ensure accuracy.

2 SCM Formulation and resolution of motion's equation for a plate2.1 Theoretical model

The geometry of the flat waveguide used in this paper and the system of axes are shown in figure 1. we consider a 2h thick plate with a wave propagating longitudinally along the z axis.



Figure 1. Geometry and axes for at waveguide. In the flat case the z axis is the phase direction of the propagating waves (normal to the plane of the wavefront). 2h is the plate thickness and the free faces of the plate are placed at y=+h and y=-h.

For the application of the spectral method, we use the Chebyshev differentiation matrices provided by Weideman and Reddy 2000[14]. Chebyshev points are used to interpolate unknown functions as follows:

$$x_j = \cos\left(\frac{(i-1)\pi}{N-1}\right), \ j = 1, \dots, N.$$
 (1)

2.2 Equation of motion

A complete description of the SCM can be found in [15]. The equations of motion for a linear elastic anisotropic homogeneous medium are :

$$\nabla_{iK} C_{KL} \nabla_{Lj}^{sym} \mathbf{u}_j = -\rho \,\,\omega^2 \,\mathbf{u}_i \tag{2}$$

Where we use the convention of summation on the indices and C_{KL} is the stiffness matrix of the medium in reduced index notation [4], u_i are the components of the vector of the field of displacement.

$$u_j = U_j(y) e^{i(kz - \omega t)}$$
; $j = x, y, z,$ (3)

By taking the faces of the plate located at $y = \pm h$, see Figure 1, and the plate is considered uncontrained at the top and bottom surfaces, the boundary conditions (BC) are as :

$$\sigma_{yy}|_{y=\pm h} = \sigma_{yx}|_{y=\pm h} = \sigma_{yz}|_{y=\pm h} = 0.$$
(4)

The expression of the stress tensor field as a function of the strain tensor field reads

$$\sigma_{ij} = C_{ijkl} S_{kl} \tag{5}$$

where C_{ijkl} is the fourth rank stiffness tensor, which relates to C_{KL} as described in [16]. And the strain tensor field, S_{ij} , in terms of the displacement vector field, u_i , is

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \to \quad S_K = \nabla_{Kj}^{sym} u_j.$$
(6)

We end up with three equations of motion and an additional six equations for the boundary conditions. The mth derivative with respect to y is approximated by the corresponding mth order Chebyshev DM.

$$\frac{\partial^{(m)}}{\partial y^{(m)}} \to D^{(m)} \coloneqq [D \ M_{Cheb}]_{N \times N}^{(m)}.$$
(7)

Each of eq. (2) becomes represented in matrix form as, for example, for the component x of displacement.

$$A_{N \times N} U_x + B_{N \times N} U_y + C_{N \times N} U_z = -\rho \omega^2 U_x.$$
(8)

A similar matrix representation emerges for each of the other components of the displacement vector field. The prefactors $A_{N\times N}$, $B_{N\times N}$ and $C_{N\times N}$ are N×N matrices formed from a linear combination of the DMs up to the second degree $(D^{(1)}, D^{(2)})$ and the identity matrix I with elastic stiffness constants, C_{KL} , as its coefficients.

It is a matrix system where the unknowns are the vectors U_j and the coefficients are the matrices $A_{N\times N}$, $B_{N\times N}$ and $C_{N\times N}$. This becomes clearer when we rearrange this system as

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}_{3N \times 3N} \begin{cases} U_x \\ U_y \\ U_z \end{cases}_{3N \times 1} = \omega^2 \begin{bmatrix} -\rho I & 0 & 0 \\ 0 & -\rho I & 0 \\ 0 & 0 & -\rho I \end{bmatrix} \begin{cases} U_x \\ U_y \\ U_z \end{cases}.$$
(9)

Or, more specifically.

$$L(k) U = \omega^2 M U, \qquad (10)$$

Where U is the vector of displacement vectors: $[U_x, U_y, U_z]^T$. The boundary conditions are then introduced; the six equations (4) are discretized and rearranged, as in [15], so

$$\sigma(k) = \begin{bmatrix} S_A & S_B & S_C \\ S_D & S_E & S_F \\ S_G & S_H & S_I \end{bmatrix} \begin{cases} U_x \\ U_y \\ U_z \end{cases} = \begin{cases} 0 \\ 0 \\ 0 \end{cases}.$$
(11)

To proceed, in Eq. (9), we replace the 1, N, (N+1), 2N, (2N+1) and 3N rows of the matrix L with those of the matrix S of Eq. (11). These lines correspond, for each component of the field of displacement vectors, to the points of the grid $y = \pm h$, i.e. lines 1 and N start with U_x evaluated at y = h and y = -h, respectively, the rows N+1 and 2N go with U_y evaluated at y = h and y = -h, and so on. Similarly, we replace the same rows of the matrix M on the right side with rows filled with zeros.

2.3 Eigenvalue problem

A reorganization of the terms in the matrix equation (10) will lead to an eigenvalue problem in terms of the wavenumber k having the expression:

$$[Q_2k^2 + Q_1k + Q_0\omega^2]U = 0 (12)$$

The matrix on the left of equation (12) is not regular and admits infinite eignvalues [17]. We use the Linear Companion Matrix Method [18] as a method of linearization, thus, the eigenvalue problem can be written in the following form:

$$(A - kB)X = 0_{6N} \,. \tag{13}$$

where A; B, and *X* are defined as :

$$A = \begin{bmatrix} Q_1 & Q_0 + M \\ I_{3N} & Z_{3N} \end{bmatrix}, \quad B = \begin{bmatrix} -Q_2 & Z_{3N} \\ Z_{3N} & I_{3N} \end{bmatrix}, \quad X = \begin{cases} kU \\ U \end{cases}$$

To ensure numerical stability, the identity matrices in the lower halves of A and B should be scaled by a constant whose magnitude is comparable to the entries in Q_i and M.

Using this formulation, a complex wavenumbers $k=k_r+ik_i$ can be solved for by fixing real ω . So we can study both propagative (real wavenumber) and non-propagative (complex wavenumber) modes.

3 Results and discussion

An example of isotropic plate is studied in this paper. The plate material is an aluminum with the characteristics cited in the table below:

Thickness	1 mm	Poisson's ratio	0.33
Young's modulus	72.4 GPa	Mass density	2770 kg/m3

 Table 1 : Characteristics of the studied plate

A Matlab programs are established based on SCM and SAFE methods to compute (k,f) dispersion curves. The symmetric (S) and the anti-symmetric modes (A) are differentiated by checking the displacement components obtained from the eigenvectors.



For the spectral method we define $N_c=2m+10$ to obtain the necessary precision (see a formal demonstration based on the convergence rates of Chebyshev series in Gottlieb and Orszag [19]), with *m* being the mode number. This condition is verified in the plotting of the wavenumber dispersion curve as a function of the frequency see figure 2. We note that the accuracy has improved from $N_c=20$ to $N_c=30$ for the first 10 modes, see figure 2 (a) and figure 2 (b), respectively.



Figure 3. (a) Evolution of relative error of A0 mode computed with SCM, with Bisection method as reference for a given two frequencies f1 < f2. (b) Running time using SCM with respect to the number of collocation points (N_c).

Figure 3 (a) shows that is a positive correlation between the frequency and the relative error. For a given number of collocation points (N_c), when the frequency increases the error does as well.

Figure 3 (b) shows the variation in running time using a computer with a processor of 2GHz and a memory (RAM) of 4 Go. The stepwise increment in k is set as 0.5. we note that the graph is closely cubic in N_c .

The Graphical representation of the wavenumber-frequency dependence computed with the SCM are exemplarily shown in Figure 4 (a) and have been found to present very good agreement with those in Figure 4 (b) computed with the SAFE method. On one hand, in Figure 4 (a), junction points 1,2,3,4 show frequency-thickness products for which the modes change in nature and become propagative, non-propagative, or attenuated. On the other hand, in figure 4 (b), the SH modes are included and the purely imaginary modes too (non-propagating highly attenuated modes).



Figure 4. Complex dispersion curves in an aluminum plate: (a) Lamb modes plotted using the SCM. (b) Lamb and SH modes plotted using SAFE, corresponding to the symmetric (in red) and antisymmetric (in bleu).



Figure 5. Wavenumber curve of nonpropagative Lamb modes: SCM (dashed lines) and SAFE (solid lines).

Very good accuracy observed at low frequencies for the non-propagative modes which remain confined in their excitation zone.

4 Conclusion

In order to evaluate the efficiency of the SCM method in plotting dispersion curves for an isotropic aluminum plate, we have established an algorithm based on the theoretical formulation presented at the beginning of this paper. The results were compared with two numerical methods based on SMM and SAFE method.

The SCM remains conceptually simpler, no need to handle special functions such as interpolation functions, easy in programming and has a high convergence rate by increasing the number of collocation points.

The results found illustrate the significant capability of the method to find all the modes without missing any and without parasitic modes, unlike zero-finding and SMM methods which require processing in the choice of the frequency-thickness step.

The ability of SCM to find complete three-dimensional solutions for dispersion curves has been demonstrated here. The plots include real, complex, and purely imaginary wavenumbers.

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